Asymptotic Behavior of Blowup Solutions for Elliptic Equations with Exponential Nonlinearity and Singular Data

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Abstract

We consider a sequence of blowup solutions of a two dimensional, second order elliptic equation with exponential nonlinearity and singular data. This equation has a rich background in physics and geometry. In a work of Bartolucci-Chen-Lin-Tarantello it is proved that the profile of the solutions differs from global solutions of a Liouville type equation only by a uniformly bounded term. The present paper improves their result and establishes an expansion of the solutions near the blowup points with a sharp error estimate.

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1 Introduction

Two dimensional semilinear elliptic equations with exponential nonlinearities arise naturally in conformal geometry and physics. The study of these equations is always

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related to their blowup phenomena. When a sequence of solutions tends to infinity near a blowup point, the asymptotic behavior of the solutions near the blowup point carries important information. In some applications it is crucial to completely understand the asymptotic behavior of blowup solutions. In this article we study the following equation:

$$\Delta u+|x|^{2\alpha}H(x)e^{u}=0,\quad \text{in } B_1\subset\mathbb{R}^2, \tag{1.1}$$

where B_1 is the unit ball in $\mathbb{R}^2$, $\alpha\in\mathbb{R}^+\setminus\mathbb{N}$, ($\mathbb{N}$ is the set of natural numbers) and $H\in C^3(\overline{B}_1)$ is a positive function. If a sequence of solutions $\{u_i\}$ tends to infinity near a point other than 0, the coefficient function $|x|^{2\alpha}H(x)$ is bounded above and below near the blowup point. This situation has been extensively studied and the asymptotic behavior of the blowup solutions is well understood (see, for example [3], [5],[13],[24],[25],[10],[36]). In this article we mainly consider a sequence of solutions $\{u_i\}$ of (1.1) such that

$$u_i(z_i)=\max_{B_1}u_i\to\infty,\quad z_i\to 0,\quad 0 \text{ is the only blowup point in } \overline{B}_1. \tag{1.2}$$

We shall describe the asymptotic profile of $\{u_i\}$ near 0 under natural assumptions on H and the oscillation of $\{u_i\}$ on ∂B_1.

The blowup analysis for (1.1) near 0 reflects the bubbling feature of a few important equations or systems of equations in physics or geometry. For example, the following mean field equation is defined on Riemann surfaces:

$$\Delta_g w+\rho\left(\frac{h(x)e^{w}}{\int_M h(x)e^{w}dV_g}-\frac{1}{|M|}\right)=4\pi\sum_{j=1}^{m}\alpha_j(\delta_{p_j}-\frac{1}{|M|}) \tag{1.3}$$

where M is a compact smooth Riemann surface without boundary, h is a positive smooth function on M, ρ is a positive constant, $|M|$ is the volume of M, Δ_g is the Laplace-Beltrami operator and $\alpha_j\delta_{p_j}$ are Dirac sources. For (1.3), the profile of blowup solutions near each p_j is exactly described by (1.1) and α_j in (1.3) plays the same role as α in (1.1). From the physical point of view, it is important to consider the case $\alpha_j>0$ in (1.3) as it is closely related to the self-dual equations in the Abelian Chern-Simons-Higgs theory (see [16],[18],[19], etc) and the Electroweak theory (see [1][23], etc). There is a considerable literature on the mean field equation (1.3) and closely related topics, we list the following as a partial list: [3][8][10][11][12][15] [17][26][27][28][29][30][31].

Another application comes from the 2-dimensional open Toda system for $SU(N+$

1). The simplest example is

$$\begin{cases} -\Delta u_1^k = 2h_1^k e^{u_1^k} - h_2^k e^{u_2^k} \\ \\ -\Delta u_2^k = 2h_2^k e^{u_2^k} - h_1^k e^{u_1^k} \end{cases} \quad \text{in} \quad B_1$$

where u_1^k, u_2^k are sequences of blowup solutions and h_1^k, h_2^k are positive, smooth functions very close to 1. Even for this simplest example, the blowup analysis is challenging because u_1^k and u_2^k may have common blowup points and the asymptotic behavior of them near their common blowup points is not yet well understood (see [20],[21], [22],[33],[30] and the references therein for recent development). The analysis for (1.1) is closely related to the Toda system and the result we prove in this work (Theorem 1.2) helps to understand the system.

In addition to the background in physics, (1.1) has a well known interpretation in geometry. Let g_0 be the Euclidean metric on B_1, then $\frac{1}{2}|x|^{2\alpha}H(x)$ is the Gauss curvature under metric $e^u g_0$. In this sense (1.1) is related to the Nirenberg problem or more generally the Kazdan-Warner problem.

When α is equal to 0, the behavior of a sequence of blowup solutions $\{u_i\}$ to

$$\Delta u_i + H_i(x)e^{u_i} = 0 \quad \text{in} \quad B_1 \tag{1.4}$$

has been extensively studied through the works of Brezis-Merle [5], Li-Shafrir[25], Li[24], Chen-Lin[10] and the references therein. In [24] Li proved that if a sequence of blowup solutions $\{u_i\}$ of (1.4) has a bounded oscillation near their blowup point, then in a neighborhood of this point $\{u_i\}$ is only $O(1)$ different from a sequence of standard bubbles appropriately scaled. Later Chen-Lin [10] and the author [36] improved Li's estimate to the sharp form by different approaches. Since the case $\alpha > 0$ is more meaningful in physics, it is important to obtain similar results for equation (1.1) when $\alpha > 0$.

In this article we address the case $\alpha \notin \mathbb{N}$. Our work is based on a result by Bartolucci-Chen-Lin-Tarantello [2] who studied (1.3) and established a result similar to Li's result for $\alpha = 0$. More specifically, let $\{u_i\}$ be a sequence of functions solving

$$\Delta u_i + |x|^{2\alpha}H_i(x)e^{u_i} = 0, \quad \text{in } B_1, \tag{1.5}$$

such that (1.2) holds. Suppose $\{u_i\}$ has bounded oscillation on ∂B_1 and a bound on the energy:

$$\begin{cases} |u_i(x) - u_i(x')| \le C_0 \quad \forall x, x' \in \partial B_1, \\ \int_{B_1} |x|^{2\alpha}H_i(x)e^{u_i} \le C_0. \end{cases} \tag{1.6}$$

Then the following result is proved in [2]:

Theorem 1.1. *(Bartolucci-Chen-Lin-Tarantello) Let $\{u_i\}$ satisfy (1.5), (1.2), (1.6) and let H_i satisfy*

$$0<\frac{1}{C_1}\le H_i(x)\le C_1,\quad \|\nabla H_i\|_{L^\infty(B_1)}\le C_1.$$

Then for $\alpha\in\mathbb{R}^+\setminus\mathbb{N}$, there exists $C>0$ such that

$$\left|u_i(x)-\log\frac{e^{u_i(0)}}{(1+\frac{H_i(0)}{8(1+\alpha)^2}e^{u_i(0)}|x|^{2\alpha+2})^2}\right|\le C\quad \textit{in } B_1.$$

Theorem 1.1 is a refinement of a result in Bartolucci-Tarantello [3], concerning the quantization phenomena for blowup solutions of (1.1) by using an improved argument based on the Pohozaev Identity. The idea of using Pohozaev's type arguments was first introduced by Bartolucci-Tarantello [3] in the analysis of the bubbling phenomena for (1.1) with $\alpha>0$.

In our main result below we assume the following on H_i:

$$0<\frac{1}{C_1}\le H_i(x)\le C_1,\quad \|H_i\|_{C^3(B_1)}\le C_1, \tag{1.7}$$

and we consider the harmonic function ψ_i solving

$$\begin{cases}\Delta\psi_i=0 \quad \text{in } B_1,\\ \\ \psi_i=u_i-\frac{1}{2\pi}\int_{\partial B_1}u_i dS\quad \text{on } \partial B_1.\end{cases} \tag{1.8}$$

Clearly ψ_i is a bounded function on B_1 and $\psi_i(0)=0$. For simplicity, we define

$$V_i(x)=H_i(x)e^{\psi_i(x)},$$

and introduce the following two constants:

$$\Lambda_1=-\frac{\pi}{V_i(0)\sin\left(\frac{\pi}{1+\alpha}\right)(1+\alpha)}\left(\frac{8(1+\alpha)^2}{V_i(0)}\right)^{\frac{1}{1+\alpha}}, \tag{1.9}$$

$$\Lambda_2=\frac{\pi}{V_i^2(0)\sin(\frac{\pi}{1+\alpha})(1+\alpha)}\left(\frac{8(1+\alpha)^2}{V_i(0)}\right)^{\frac{1}{1+\alpha}}. \tag{1.10}$$

Note that $\psi_i(0)=0$ implies the following:

$$V_i(0)=H_i(0),\quad \nabla V_i(0)=\nabla H_i(0)+H_i(0)\nabla\psi_i(0)$$
$$\Delta V_i(0)=\Delta H_i(0)+2\nabla H_i(0)\cdot\nabla\psi_i(0)+H_i(0)|\nabla\psi_i(0)|^2.$$

Our main theorem improves Theorem 1.1 as follows:

Theorem 1.2. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Suppose u_i satisfies (1.5), (1.2), (1.6) and (1.7). Then in B_1 there holds*

$$\begin{aligned} u_i(x) &= \log \frac{e^{u_i(0)}}{(1+\frac{V_i(0)}{8(1+\alpha)^2}e^{u_i(0)}|x|^{2\alpha+2})^2} + \psi_i(x) \\ &\quad - \frac{2(1+\alpha)}{\alpha V_i(0)} \frac{\nabla V_i(0)\cdot x}{1+\frac{V_i(0)}{8(1+\alpha)^2}e^{u_i(0)}|x|^{2\alpha+2}} \\ &\quad + \Big(\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2\Big) \log\Big(2 + e^{\frac{u_i(0)}{2(1+\alpha)}}|x|\Big) e^{-\frac{u_i(0)}{1+\alpha}} + O(e^{-\frac{u_i(0)}{1+\alpha}}), \end{aligned}$$

where ψ_i, Λ_1, Λ_2 are defined by (1.8), (1.9) and (1.10), respectively.

Note that we use $O(e^{-\frac{u_i(0)}{1+\alpha}})$ to denote a smooth function in B_1 whose C^3 norm is bounded by $C(\alpha, C_0, C_1)e^{-\frac{u_i(0)}{1+\alpha}}$.

Theorem 1.2 corresponds to the results of Chen-Lin [10] and the author [36] for the case $\alpha = 0$. One important application of getting sharp blowup estimates is to derive a degree counting formula for the mean field equation (1.3). When the right hand side of (1.3) is 0, Chen-Lin [11] established a degree counting formula in terms of the genus of the Riemann surface using the sharp estimate in [10]. Another way of counting the degree has been proved by Malchiodi [32]. We expect to use Theorem 1.2 to derive a degree counting formula for the general mean field equation (1.3) in a forthcoming paper.

The assumption $\alpha \notin \mathbb{N}$ is essential to Theorem 1.2. When (1.5) is compared with the case $\alpha = 0$, many important features are different. For the latter case, the method of moving spheres and the Pohozaev's type arguments are very effective. However, these well known methods do not seem to be useful for the former case. Instead we mainly use the potential theory iteratively to obtain the sharp estimate. Heuristically, the main difference between the two cases is that when $\alpha > 0 (\alpha \notin \mathbb{N})$, the linearized operator of (1.5) along a standard bubble is "invertible". This invertibility forces the maximum points of blowup solutions to be very close to 0 (see Corollary 2.1). While for the $\alpha = 0$ case, this invertibility is lost and it is important to obtain the sharp vanishing rate of the gradient of some coefficient functions, see [36] for details.

We also note that the assumption $\alpha \notin \mathbb{N}$ cannot be removed in general. In fact even Theorem 1.1 may not hold for $\alpha \in \mathbb{N}$ (see [2] [35]). However the case $\alpha \in \mathbb{N}$ has important applications (see [34]) and needs to be better understood. A theorem similar to Theorem 1.2 in this case, even with more assumptions, should still be of much interest.

The structure of this paper is as follows: In section 2 we prove Theorem 1.2. Our proof is based on Theorem 1.1. Then in the appendix we include some detailed estimates.

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2 Proof of Theorem 1.2

We shall always use C to denote a constant depending on α, C_0, C_1 only, unless we specify otherwise.

2.1 A uniqueness lemma

Lemma 2.1. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, ϕ be a C^2 function that verifies*

$$\begin{cases} \Delta\phi + |x|^{2\alpha} e^{U_\alpha}\phi = 0 \qquad \text{in } \mathbb{R}^2, \\ \\ \phi(0) = 0, \quad |\phi(x)| \le C(1+|x|)^\tau \quad x \in \mathbb{R}^2. \end{cases}$$

where

$$U_\alpha(x) = \log \frac{8(\alpha+1)^2}{(1+|x|^{2\alpha+2})^2} \qquad \text{and} \qquad \tau \in [0,1).$$

Then $\phi \equiv 0$.

Proof of Lemma 2.1: Let $k \ge 1$ be an integer. We define

$$\phi_k(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi(r\cos\theta, r\sin\theta)\cos(k\theta)d\theta.$$

Then ϕ_k satisfies

$$\begin{cases} \phi_k''(r) + \frac{1}{r}\phi_k'(r) + (r^{2\alpha}e^{U_\alpha} - \frac{k^2}{r^2})\phi_k(r) = 0, \quad 0 < r < \infty, \\ \\ \lim_{r\to 0}\phi_k(r) = 0, \quad |\phi_k(r)| \le C(1+r)^\tau, \quad r > 0. \end{cases} \tag{2.1}$$

Putting $r = e^t$ and using Theorem 8.1 of [14, Theorem 8.1, p.92] one can see there are two fundamental solutions of (2.1) that behave like those of

$$\tilde{\phi}''(r) + \frac{1}{r}\tilde{\phi}'(r) - \frac{k^2}{r^2}\tilde{\phi}(r) = 0$$

as $r \to \infty$. (Note that the $\int_0^\infty |V'(t)|dt < \infty$ in the statement of Theorem 8.1 from [14] corresponds to $\int_0^\infty r^{2\alpha+2} e^{U_\alpha} dr < \infty$, which is readily verified.)

Let us denote by $\phi_{k,1}$ and $\phi_{k,2}$ the two fundamental solutions of (2.1) so that $\phi_{k,1} \sim r^k$, $\phi_{k,2} \sim r^{-k}$ as $r \to \infty$. Since $\phi_k = c_1\phi_{k,1} + c_2\phi_{k,2}$ and $|\phi_k|$ grows no faster than r^τ as $r \to \infty$, we conclude that $c_1 = 0$. Hence $|\phi_k| \sim r^{-k}$ for r large. On the other hand, when r is close to 0, the term $r^{2\alpha}e^{U_\alpha}$ is a perturbation again, which means there are two fundamental solutions, say, $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$, comparable to r^k and r^{-k} as $r \to 0+$, respectively. Since $\phi_k(0) = 0$, we know that $|\phi_k(r)| \sim r^k$ as $r \to 0+$.

We claim that $\phi_k \equiv 0$ for all $k \geq 1$. To see this, let $f_k(s) = \phi_k(s^{\frac{1}{1+\alpha}})$, then we have

$$f_k''(s) + \frac{1}{s} f_k'(s) + \left(\frac{8}{(1+s^2)^2} - \frac{k^2}{(1+\alpha)^2 s^2} \right) f_k(s) = 0, \quad 0 < s < \infty. \tag{2.2}$$

Using $\phi_k = O(r^{-k})$ at infinity and $\phi_k = O(r^k)$ at 0 we conclude that

$$f_k(s) = O(s^{-\frac{k}{1+\alpha}}) \quad \text{at} \quad \infty, \qquad f_k(s) = O(s^{\frac{k}{1+\alpha}}) \quad \text{at} \quad 0. \tag{2.3}$$

Let $\delta_1 = \frac{k}{1+\alpha}$. Since $\alpha \notin \mathbb{N}$ we see clearly that $\delta_1 \neq 1$. Then by direct computation we verify that the following two functions are two fundamental solutions of (2.2):

$$\begin{aligned} f_{11}(s) &= \frac{(\delta_1 + 1)s^{\delta_1} + (\delta_1 - 1)s^{\delta_1 + 2}}{1+s^2}, \\ f_{12}(s) &= \frac{(\delta_1 + 1)s^{2-\delta_1} + (\delta_1 - 1)s^{-\delta_1}}{1+s^2}. \end{aligned} \tag{2.4}$$

Consequently $f_k = c_1 f_{11} + c_2 f_{12}$ where c_1 and c_2 are two constants. From (2.3) we see that $c_1 = 0$ because otherwise $|f_k| \sim s^{\delta_1}$ at ∞. Similarly by observing the behavior of f_k at 0 we have $c_2 = 0$ because otherwise $|f_k| \sim s^{-\delta_1}$ near 0, which is a contradiction to (2.3) again. So $f_k \equiv 0$ for all $k \geq 1$. This is equivalent to $\phi_k \equiv 0$ for all $k \geq 1$. The same argument also shows that the projection of ϕ along $\sin k\theta$ ($\forall k \geq 1$) is 0. Finally, from $\phi(0) = 0$ we conclude $\phi(x) \equiv 0$.$\Box$

2.2 Initial refinement

In this subsection, we give the first improvement of Theorem 1.1. The main result in this subsection is Proposition 2.1.

Let $\delta_i = e^{-\frac{u_i(0)}{2+2\alpha}}$ and we denote by z_i the maximum point of u_i. It is proved in [2] that $\delta_i^{-1} z_i \to 0$. Note that we use $u_i(0)$ to define δ_i because $u_i(0)/u_i(z_i) = 1 + \circ(1)$. From the definition of ψ_i, $u_i - \psi_i$ has no oscillation on ∂B_1 and it satisfies

$$\Delta(u_i - \psi_i) + |x|^{2\alpha} V_i(x) e^{(u_i - \psi_i)(x)} = 0, \quad \text{in } B_1$$

where $V_i(x) = H_i(x)e^{\psi_i(x)}$ is defined as before. Without loss of generality we assume

$$V_i(0) = H_i(0) \to 8(\alpha+1)^2.$$

Let x_i be the maximum point of $u_i - \psi_i$, then we still have $\delta_i^{-1}x_i \to 0$ by the argument in [2]. Now we define v_i as

$$v_i(y) = u_i(\delta_i y) - \psi_i(\delta_i y) - u_i(0), \quad y \in \Omega_i := B(0, \delta_i^{-1}).$$

We list some properties of v_i implied by its definition:

$$\begin{cases} \Delta v_i(y) + |y|^{2\alpha} V_i(\delta_i y) e^{v_i(y)} = 0, \quad y \in \Omega_i := B(0, \delta_i^{-1}), \\ v_i(0) = 0, \quad y_i := \delta_i^{-1} x_i \to 0. \\ v_i(y) \to (-2)\log(1+|y|^{2\alpha+2}) \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2). \\ v_i(y_1) = v_i(y_2), \quad \forall y_1, y_2 \in \partial\Omega_i := \partial B(0, \delta_i^{-1}). \end{cases}$$

Note that in the second equation above we use y_i to denote the maximum point of v_i. Let

$$U_i(y) = (-2)\log\left(1 + \frac{V_i(0)}{8(\alpha+1)^2}|y|^{2\alpha+2}\right)$$

be a standard bubble which satisfies

$$\Delta U_i + |y|^{2\alpha} V_i(0) e^{U_i(y)} = 0, \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

Then the conclusion of Theorem 1.1 can be written as:

$$|v_i(y) - U_i(y)| \le C, \quad |y| \le \delta_i^{-1}.$$

Let $w_i(y) = v_i(y) - U_i(y)$, then the following proposition is the first improvement of Theorem 1.1:

Proposition 2.1. *For any $\epsilon \in (0, \frac{1}{2})$, there exists $C(\epsilon) > 0$ such that for all large i,*

$$|w_i(y)| \le C\delta_i(1+|y|)^{\epsilon}, \quad \text{in } \Omega_i.$$

Proof of Proposition 2.1: We write the equation for w_i as

$$\begin{cases} \Delta w_i + r^{2\alpha} V_i(0) e^{\xi_i} w_i = O(\delta_i)(1+r)^{-3-2\alpha}, \\ w_i(0) = 0, \quad |w_i(y)| \le C, \quad y \in \Omega_i, \quad w_i|_{\partial\Omega_i} = \tilde{a}_i, \end{cases}$$

where ξ_i is obtained by the mean value theorem. From Theorem 1.1 we immediately have $\tilde{a}_i = O(1)$. Let

$$\tilde{\Lambda}_i = \max \frac{|w_i(y)|}{\delta_i(1+|y|)^{\epsilon}} \quad y \in \bar{\Omega}_i.$$

Our goal is to show $\tilde{\Lambda}_i = O(1)$. We prove this by a contradiction. Suppose $\tilde{\Lambda}_i \to \infty$, then we use $\tilde{y}_i$ to denote a point where $\tilde{\Lambda}_i$ is assumed. Let

$$\bar{w}_i(y) = \frac{w_i(y)}{\tilde{\Lambda}_i \delta_i (1+|\tilde{y}_i|)^{\epsilon}}.$$

It readily follows from the definition of $\tilde{\Lambda}_i$ that $|\bar{w}_i(y)| \le \frac{(1+|y|)^{\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}}$, which means $\bar{w}_i$ is uniformly bounded over any fixed compact subset of $\mathbb{R}^2$. Therefore we conclude that, along a subsequence, $\bar{w}_i$ converges in $C^2_{loc}(\mathbb{R}^2)$ to a solution w of

$$\begin{cases} \Delta w + r^{2\alpha} e^{U_\alpha} w = 0, \quad \mathbb{R}^2, \\ \\ w(0) = 0, \quad |w(y)| \le C(1+|y|)^{\epsilon}. \end{cases}$$

If $\tilde{y}_i$ converges to $y_0 \in \mathbb{R}^2$, we have $|w(y_0)| = 1$ by continuity. However this is impossible because an application of Lemma 2.1 shows that $w \equiv 0$. Therefore the only case to consider is $\tilde{y}_i \to \infty$.

It follows from $|\bar{w}_i(\tilde{y}_i)| = 1$ and the Green's representation formula that

$$\begin{aligned} \pm 1 = \bar{w}_i(\tilde{y}_i) = & \int_{\Omega_i} G(\tilde{y}_i, \eta) \Big\{ |\eta|^{2\alpha} V_i(0) e^{\xi_i(\eta)} \frac{w_i(\eta)}{\tilde{\Lambda}_i \delta_i (1+|\eta|)^{\epsilon}} \frac{(1+|\eta|)^{\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}} \\ & + \frac{O(1)(1+|\eta|^{-3-2\alpha})}{\tilde{\Lambda}_i (1+|\tilde{y}_i|)^{\epsilon}} \Big\} d\eta - \int_{\partial\Omega_i} \frac{\partial G}{\partial \nu}(\tilde{y}_i, \eta) \frac{\tilde{a}_i}{\tilde{\Lambda}_i \delta_i (1+|\tilde{y}_i|)^{\epsilon}} dS \end{aligned} \tag{2.6}$$

where G is the Green's function over Ω_i with respect to the Dirichlet boundary condition. Recall that the Green's function over Ω_i is

$$G(y, \eta) = -\frac{1}{2\pi} \log|y - \eta| + \frac{1}{2\pi} \log\left(\frac{|y|}{\delta_i^{-1}} \Big| \frac{\delta_i^{-2} y}{|y|^2} - \eta \Big| \right).$$

Since $\bar{w}_i(0) = 0$, the Green's representation formula gives

$$\begin{aligned} 0 = & \int_{\Omega_i} G(0, \eta) \Big\{ |\eta|^{2\alpha} V_i(0) e^{\xi_i(\eta)} \frac{w_i(\eta)}{\tilde{\Lambda}_i \delta_i (1+|\eta|)^{\epsilon}} \frac{(1+|\eta|)^{\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}} \\ & + \frac{O(1)(1+|\eta|)^{-3-2\alpha}}{\tilde{\Lambda}_i (1+|\tilde{y}_i|)^{\epsilon}} \Big\} d\eta - \int_{\partial\Omega_i} \frac{\partial G}{\partial \nu}(0, \eta) \frac{\tilde{a}_i}{\tilde{\Lambda}_i \delta_i (1+|\tilde{y}_i|)^{\epsilon}} dS. \end{aligned} \tag{2.7}$$

To deal with the two boundary integral terms in (2.6) and (2.7), we observe that

$$\int_{\partial\Omega_i}\left(\frac{\partial G}{\partial\nu}(\tilde{y}_i,\eta)-\frac{\partial G}{\partial\nu}(0,\eta)\right)\frac{\tilde{a}_i}{\tilde{\Lambda}_i\delta_i(1+|\tilde{y}_i|)^{\epsilon}}dS=0.$$

This equality follows by using the well known identity

$$\int_{\partial\Omega_i}\partial_\nu G(\xi,\eta)dS=-1,\quad \forall\xi\in\Omega_i.$$

From (2.6) and (2.7) we have

$$1\le\int_{\Omega_i}|G(\tilde{y}_i,\eta)-G(0,\eta)|\left(\frac{(1+|\eta|)^{-4-2\alpha+\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}}+\circ(1)\frac{(1+|\eta|)^{-3-2\alpha}}{(1+|\tilde{y}_i|)^{\epsilon}}\right)d\eta. \tag{2.8}$$

Note that in the above we used

$$|\frac{w_i(\eta)}{\tilde{\Lambda}_i\delta_i(1+|\eta|)^{\epsilon}}|\le 1,\quad e^{\xi_i(\eta)}\le C(1+|\eta|)^{-4-4\alpha},\quad \tilde{\Lambda}_i\to\infty.$$

To get a contradiction to (2.8) we only need to show the following:

$$\int_{\Omega_i}|G(\tilde{y}_i,\eta)-G(0,\eta)|\left(\frac{(1+|\eta|)^{-4-2\alpha+\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}}+\circ(1)\frac{(1+|\eta|)^{-3-2\alpha}}{(1+|\tilde{y}_i|)^{\epsilon}}\right)d\eta=\circ(1). \tag{2.9}$$

We consider two cases: If $|\tilde{y}_i|=\circ(1)\delta_i^{-1}$, $G(\tilde{y}_i,\eta)$ can be written as

$$G(\tilde{y}_i,\eta)=-\frac{1}{2\pi}\log|\tilde{y}_i-\eta|+\frac{1}{2\pi}\log\delta_i^{-1}+\circ(1).$$

In this case it is enough to show

$$\int_{\Omega_i}\left|\log\frac{|\tilde{y}_i-\eta|}{|\eta|}\right|\left(\frac{(1+|\eta|)^{-4-2\alpha+\epsilon}}{(1+|\tilde{y}_i|)^{\epsilon}}+\circ(1)\frac{(1+|\eta|)^{-3-2\alpha}}{(1+|\tilde{y}_i|)^{\epsilon}}\right)d\eta=\circ(1),$$

which follows from standard elementary estimates.

Finally we consider the case $|\tilde{y}_i|\sim\delta_i^{-1}$. For the Green's function we use

$$|G(\tilde{y}_i,\eta)-G(0,\eta)|\le C(\log(1+|\eta|)+\log\delta_i^{-1}),\quad C\text{ universal }.$$

Then it is easy to obtain (2.9) by elementary estimates. Proposition 2.1 is established. $\square$

From Proposition 2.1 we obtain an estimate on x_i more precise than $|x_i|=\circ(\delta_i)$ (Recall that x_i is the place where the maximum of $u_i-\psi_i$ occurs, y_i is the point where the maximum of v_i is attained.).

Corollary 2.1.

$$|y_i| = |\delta_i^{-1}x_i| = O(\delta_i^{1/(2\alpha+1)}). \tag{2.10}$$

Proof of Corollary 2.1: Using Proposition 2.1 and standard elliptic estimates we have

$$|v_i(y) - U_i(y)| \le C\delta_i|y|, \quad |y| \le 10$$

and

$$|\nabla v_i(y) - \nabla U_i(y)| \le C\delta_i, \quad |y| \le 10.$$

Therefore $|\nabla U_i(y_i)| = O(\delta_i)$ because $\nabla v_i(y_i) = 0$. Then (2.10) follows from the definition of U_i. $\Box$

Remark 2.1. *Recall that z_i is the place where the maximum of u_i is attained. For z_i we also have $z_i = O(\delta_i^{\frac{2\alpha+2}{2\alpha+1}})$ by the same argument.*

2.3 Further refinement of the expansion

U_i is the first term in the expansion of v_i. To determine the second term in the expansion we need a radial function g_i that satisfies the following:

$$\begin{cases} g_i''(r) + \frac{1}{r}g_i'(r) + (r^{2\alpha}V_i(0)e^{U_i(r)} - \frac{1}{r^2})g_i(r) = -r^{2\alpha+1}e^{U_i(r)}, \quad 0 < r < \infty, \\ \lim_{r\to 0+} g_i(r) = \lim_{r\to\infty} g_i(r) = 0, \quad |g_i(r)| \le C\frac{r}{1+r^2}. \end{cases} \tag{2.11}$$

By direct computation one checks that the following expression verifies the above:

$$g_i(r) = -\frac{2(1+\alpha)}{\alpha V_i(0)}\frac{r}{1 + \frac{V_i(0)}{8(1+\alpha)^2}r^{2\alpha+2}}. \tag{2.12}$$

Let

$$\phi_i(y) = g_i(r)\delta_i\sum_{j=1}^{2}\partial_j V_i(0)\theta_j, \quad \theta_j = y_j/r. \tag{2.13}$$

Then ϕ_i satisfies

$$\Delta\phi_i + |y|^{2\alpha}V_i(0)e^{U_i(y)}\phi_i = -\sum_{j=1}^{2}\delta_i\partial_j V_i(0)y_j|y|^{2\alpha}e^{U_i(y)}, \quad \text{in } \Omega_i. \tag{2.14}$$

We claim that ϕ_i is the second term in the expansion of v_i. This is verified in the following

Proposition 2.2. *For any $\epsilon\in(0,\frac{1}{2})$, there is $C>0$ depending only on C_0,C_1,α,ϵ such that*

$$\begin{aligned}&|v_i(y)-(U_i+\phi_i)(y)|\le C\delta_i^2(1+|y|)^{\epsilon},\\ &|\nabla v_i(y)-\nabla(U_i+\phi_i)(y)|\le C\delta_i^2(1+|y|)^{\epsilon-1},\qquad in\ \Omega_i.\end{aligned}$$

Proof of Proposition 2.2: Let $b_i=v_i-U_i-\phi_i$. Then b_i satisfies $b_i(0)=0$ and

$$\Delta b_i+r^{2\alpha}V_i(\delta_i y)e^{\xi_i}b_i=r^{2\alpha}(V_i(0)e^{U_i}-V_i(\delta_i y)e^{\xi_i})\phi_i+O(\delta_i^2)r^{2\alpha+2}e^{U_i}.$$

By using Proposition 2.1 and (2.11) in the estimate of the right hand side of the above, we have

$$\Delta b_i+r^{2\alpha}V_i(\delta_i y)e^{\xi_i}b_i=O(\delta_i^2)(1+r)^{-2-2\alpha}.$$

Proposition 2.2 follows from the same argument as in Proposition 2.1. □.

Remark 2.2. *The essential difference between the $\alpha\notin\mathbb{N}$ case and the $\alpha=0$ case lies in the fundamental solutions of the equation in (2.11). The two sets of fundamental solutions behave very differently. As a result, for $\alpha\notin\mathbb{N}$, we have the presence of ϕ_i and the maximum point of v_i very close to the singularity 0 (see Corollary 2.1). On the other hand, for the $\alpha=0$ case, it is crucial to obtain the vanishing rate of $\nabla V_i(0)$ from the Pohozaev Identity (see [36]).*

Let $\theta_j:=\frac{y_j}{r}$, $j=1,2$, we list the following well known identities for convenience.

$$-\frac{d^2}{d\theta^2}(\theta_1\theta_2)=4\theta_1\theta_2,\quad -\frac{d^2}{d\theta^2}(\theta_j^2-\frac{1}{2})=4(\theta_j^2-\frac{1}{2}),\quad j=1,2. \tag{2.15}$$

Now we rewrite the equation for v_i using Proposition 2.2. Let $b_i=v_i-U_i-\phi_i$, then we have

$$\Delta(U_i+\phi_i+b_i)+r^{2\alpha}(V_i(0)+\delta_i\nabla V_i(0)\cdot y+F_1+O(\delta_i^3r^3))e^{U_i+\phi_i+b_i}=0 \tag{2.16}$$

where

$$\begin{aligned}F_1&=\delta_i^2(\frac{1}{2}\partial_{11}V_i(0)y_1^2+\frac{1}{2}\partial_{22}V_i(0)y_2^2+\partial_{12}V_i(0)y_1y_2)\\ &=\delta_i^2\left(\frac{1}{2}\partial_{11}V_i(0)(y_1^2-\frac{r^2}{2})+\frac{1}{2}\partial_{22}V_i(0)(y_2^2-\frac{r^2}{2})+\partial_{12}V_i(0)y_1y_2\right)\\ &\quad+\frac{1}{4}\delta_i^2r^2\Delta V_i(0)=F_{11}+F_{12}\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
F_{11} &= \delta_i^2 r^2 (\frac{1}{2}\partial_{11} V_i(0)(\theta_1^2 - \frac{1}{2}) + \frac{1}{2}\partial_{22} V_i(0)((\theta_2^2 - \frac{1}{2}) + \partial_{12} V_i(0)\theta_1\theta_2) \\
F_{12} &= \frac{1}{4}\delta_i^2 r^2 \Delta V_i(0) \qquad (2.18)
\end{aligned}
$$

By using Proposition 2.2 we have $|b_i| = O(\delta_i^2) r^\epsilon$ and $b_i + \phi_i = O(\delta_i (1+r)^{-\frac{1}{2}})$. Since $e^{U_i} = O((1+r)^{-4-4\alpha})$ we conclude that

$$
\begin{aligned}
e^{U_i+\phi_i+b_i} &= e^{U_i}(1 + \phi_i + b_i + \frac{1}{2}(\phi_i + b_i)^2 + O(\delta_i^3) r^{-\frac{3}{2}}) \\
&= e^{U_i} + e^{U_i}\phi_i + e^{U_i} b_i + \frac{1}{2} e^{U_i}\phi_i^2 + O(\delta_i^3)(1+r)^{-5-4\alpha+2\epsilon}.
\end{aligned}
$$

With this expression (2.16) is reduced to

$$
\begin{aligned}
&\Delta(U_i + \phi_i + b_i) + r^{2\alpha}\Big(V_i(0) + \delta_i \nabla V_i(0)\cdot y + F_1 + O(\delta_i^3 r^3)\Big) \\
&\cdot\Big(e^{U_i} + e^{U_i}\phi_i + e^{U_i} b_i + \frac{1}{2} e^{U_i}\phi_i^2 + O(\delta_i^3 (1+r)^{-5-4\alpha+2\epsilon})\Big) = 0.
\end{aligned}
$$

The last term on the left hand side of the above is

$$
\begin{aligned}
&r^{2\alpha}\Big(V_i(0)e^{U_i} + V_i(0)e^{U_i}\phi_i + V_i(0)e^{U_i} b_i + \frac{1}{2}V_i(0)e^{U_i}\phi_i^2 + \delta_i \nabla V_i(0)\cdot y e^{U_i} \\
&+\delta_i \nabla V_i(0)\cdot y e^{U_i}\phi_i + F_1 e^{U_i} + O(\delta_i^3)(1+r)^{-1-4\alpha}\Big).
\end{aligned}
$$

By combining (2.14), (2.5), (2.17) we can write the equation for b_i as

$$
\begin{aligned}
&\Delta b_i + r^{2\alpha} V_i(0) e^{U_i} b_i + \frac{V_i(0)}{2} r^{2\alpha} e^{U_i}\phi_i^2 + \delta_i \nabla V_i(0)\cdot y e^{U_i}\phi_i r^{2\alpha} \\
&+\frac{r^{2+2\alpha}}{4}\Delta V_i(0)\delta_i^2 e^{U_i} + r^{2\alpha} F_{11} e^{U_i} + O(\delta_i^3)(1+r)^{-1-2\alpha} = 0. \qquad (2.19)
\end{aligned}
$$

Without loss of generality we assume $\nabla V_i(0) = |\nabla V_i(0)| e_1$. Then the sum of the

third and fourth term on the left hand side of (2.19) is

$$\begin{aligned}
& \frac{V_i(0)}{2} r^{2\alpha} e^{U_i} \phi_i^2 + \delta_i r^{2\alpha} \nabla V_i(0) \cdot y e^{U_i} \phi_i \\
= \; & \frac{V_i(0)}{2} r^{2\alpha} e^{U_i} g_i^2(r) \delta_i^2 |\nabla V_i(0)|^2 \theta_1^2 + \delta_i^2 g_i(r) |\nabla V_i(0)|^2 r^{1+2\alpha} \theta_1^2 e^{U_i} \\
= \; & \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \theta_1^2 \Big(\frac{V_i(0)}{2} g_i^2(r) + g_i(r) r\Big) \\
= \; & \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \Big(\theta_1^2 - \frac{1}{2}\Big)\Big(\frac{V_i(0)}{2} g_i^2(r) + g_i(r) r\Big) \\
& \quad + \frac{1}{2} \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \Big(\frac{V_i(0)}{2} g_i^2(r) + g_i(r) r\Big) \\
= \; & C_{11} + C_{12}.
\end{aligned}$$

For C_{11} and $r^{2\alpha} F_{11} e^{U_i}$ we can find c_i that satisfies

$$\begin{cases} \Delta c_i + r^{2\alpha} V_i(0) e^{U_i} c_i + C_{11} + r^{2\alpha} F_{11} e^{U_i} = 0, \quad 0 < r < \delta_i^{-1} \\ |c_i(x)| \le C \delta_i^2 \frac{r^2}{(1+r)^3}, \quad 0 < r < \delta_i^{-1}. \end{cases} \tag{2.20}$$

The existence of c_i and its estimate are established in the appendix. Let $d_i = b_i - c_i$. Then the equation for d_i is

$$\Delta d_i + r^{2\alpha} V_i(0) e^{U_i} d_i + E_i + O(\delta_i^3 (1+r)^{-1-2\alpha}) = 0. \tag{2.21}$$

where

$$E_i = \frac{r^{2+2\alpha}}{4} \Delta V_i(0) \delta_i^2 e^{U_i} + \frac{1}{2} \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \Big(\frac{V_i(0)}{2} g_i^2(r) + g_i(r) r\Big).$$

It follows from the definition of d_i, Proposition 2.2 and (2.20), that

$$d_i(0) = 0, \quad |d_i(y)| \le C \delta_i^2 (1+r)^{\epsilon}, \quad r \le \delta_i^{-1}. \tag{2.22}$$

Also, it is implied by (2.21) and (2.22) that

$$\int_{\Omega_i} \Delta d_i = O(\delta_i^2). \tag{2.23}$$

In the following proposition we evaluate the value of d_i on $\partial \Omega_i$:

Proposition 2.3.

$$d_i = (\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2) \delta_i^2 \log \delta_i^{-1} + O(\delta_i^2) \quad on \quad \partial \Omega_i.$$

Proof of Proposition 2.3: Let

$$f_i(y)=\frac{1-a_ir^{2\alpha+2}}{1+a_ir^{2\alpha+2}},\quad a_i=\frac{V_i(0)}{8(1+\alpha)^2}.$$

Direct computation shows

$$\Delta f_i(y)+r^{2\alpha}V_i(0)e^{U_i}f_i(y)=0,\quad \text{in } \mathbb{R}^2. \tag{2.24}$$

Also it is straightforward to verify that

$$f_i(y)=-1+O(\delta_i^{2\alpha+2}),\quad \frac{\partial f_i}{\partial r}=O(\delta_i^{3+2\alpha})\quad \text{on } \partial\Omega_i. \tag{2.25}$$

A direct consequence of (2.21), (2.24) and the Green's formula is

$$\int_{\partial\Omega_i}(\frac{\partial f_i}{\partial\nu}d_i-\frac{\partial d_i}{\partial\nu}f_i)=\int_{\Omega_i}E_if_i+O(\delta_i^{2+2\alpha}). \tag{2.26}$$

From (2.25) we have

$$\int_{\partial\Omega_i}\frac{\partial f_i}{\partial\nu}d_i=O(\delta_i^{2+2\alpha}). \tag{2.27}$$

Since f_i is a radial function, we have, by (2.23) and (2.25),

$$\int_{\partial\Omega_i}\frac{\partial d_i}{\partial\nu}f_i=(\int_{\Omega_i}\Delta d_i)f_i|_{\partial\Omega_i}=-\int_{\Omega_i}\Delta d_i+O(\delta_i^{2+2\alpha}). \tag{2.28}$$

So we conclude from (2.26), (2.27) and (2.28) that

$$\int_{\Omega_i}\Delta d_i=\int_{\Omega_i}E_if_i+O(\delta_i^{2+2\alpha}). \tag{2.29}$$

On the other hand, from $d_i(0)=0$ and (2.21) we have

$$0=\int_{\Omega_i}G(0,\eta)(|\eta|^{2\alpha}V_i(0)e^{U_i(\eta)}d_i(\eta)+E_i(\eta))+d_i|_{\partial\Omega_i}+O(\delta_i^2).$$

Since

$$G(0,\eta)=-\frac{1}{2\pi}\log|\eta|+\frac{1}{2\pi}\log\delta_i^{-1}$$

and

$$\int_{\Omega_i}\log|\eta|(|\eta|^{2\alpha}V_i(0)e^{U_i(\eta)}d_i(\eta)+E_i(\eta))=O(\delta_i^2),$$

we obtain from elementary estimates

$$\begin{aligned} d_i|_{\partial\Omega_i} &= \frac{1}{2\pi}\log(\delta_i^{-1})\int_{\Omega_i} E_i f_i + O(\delta_i^2) \\ &= (\log\delta_i^{-1})\delta_i^2(\Lambda_1\Delta V_i(0)+\Lambda_2|\nabla V_i(0)|^2)+O(\delta_i^2). \end{aligned}$$

Proposition 2.3 is established. □

We now finish the proof of Theorem 1.2 by a standard application of the maximum principle. Let

$$M_i(y) = (\Lambda_1\Delta V_i(0)+\Lambda_2|\nabla V_i(0)|^2)\delta_i^2\log|y| + \delta_i^2 M(1-r^{-\alpha})$$

where M is a large number to be determined. Thanks to (2.21),(2.22) and the estimate of E_i, the equation for d_i can be written as

$$\Delta d_i = O(\delta_i^2)(1+r)^{-2-2\alpha}, \quad \text{in } \Omega_i.$$

By choosing M large enough we have $M_i(y) > d_i(y)$ on ∂B_2 and $\partial\Omega_i$. On the other hand

$$\Delta M_i(y) = -M\alpha^2\delta_i^2 r^{-2-\alpha} < \Delta d_i(y) \quad 2<r<\delta_i^{-1}.$$

Therefore by the maximum principle

$$d_i(y) \le M_i(y) \quad 2<r<\delta_i^{-1}.$$

Similarly we can also show

$$d_i(y) \ge (\Lambda_1\Delta V_i(0)+\Lambda_2|\nabla V_i(0)|^2)\delta_i^2\log|y| - \delta_i^2 M(1-r^{-\alpha}) \quad 2<r<\delta_i^{-1}$$

for M large. Theorem 1.2 is established. □

3 Appendix

In this section we prove the existence of c_i that satisfies (2.20). Recall that

$$F_{11} = \delta_i^2 r^2(\frac{1}{2}\partial_{11}V_i(0)(\theta_1^2-\frac{1}{2})+\frac{1}{2}\partial_{22}V_i(0)(\theta_2^2-\frac{1}{2})+\partial_{12}V_i(0)\theta_1\theta_2)$$

$$C_{11} = \delta_i^2 r^{2\alpha}e^{U_i}|\nabla V_i(0)|^2(\theta_1^2-\frac{1}{2})(\frac{V_i(0)}{2}g_i^2(r)+g_i(r)r).$$

The three terms in $r^{2\alpha}F_{11}e^{U_i}$ and C_{11} can all be written in the form $\delta_i^2 f(\theta)Q_i(r)$ where $f(\theta)$ is one of the spherical harmonics ($\theta_1^2-\frac{1}{2}$, $\theta_2^2-\frac{1}{2}$, $\theta_1\theta_2$) and $Q_i(r)$ is a radial function that satisfies

$$|Q_i(r)| \le C\frac{r^{2+2\alpha}}{(1+a_i r^{2+2\alpha})^2}.$$

If we can find a function $h_1(r)$ that solves

$$\begin{cases} h_1''(r)+\frac{1}{r}h_1'(r)+(r^{2\alpha}V_i(0)e^{U_i}-\frac{4}{r^2})h_1(r)=-Q_i(r), \quad 0<r<\delta_i^{-1} \\ |h_1(r)|\le C\frac{r^2}{(1+r)^3}, \quad 0<r<\delta_i^{-1}, \end{cases} \tag{3.1}$$

then by (2.15) $h_1(r)f(\theta)$ solves

$$(\Delta+r^{2\alpha}V_i(0)e^{U_i})(f(\theta)h_1(r))+f(\theta)Q_i(r)=0, \quad \Omega_i.$$

Consequently c_i is the sum of four such functions. So all we need to establish is (3.1). Let

$$r=(\sqrt{\frac{8(\alpha+1)^2}{V_i(0)}}s)^{\frac{1}{1+\alpha}} \quad \text{and } f_1(s)=h_1\Big((\sqrt{\frac{8(\alpha+1)^2}{V_i(0)}}s)^{\frac{1}{1+\alpha}}\Big).$$

Then the equation for $f_1(s)$ is

$$f_1''(s)+\frac{1}{s}f_1'(s)+(\frac{8}{(1+s^2)^2}-\frac{4\delta^2}{s^2})f_1(s)=l_1(s), \quad 0<s<\infty,$$

where $\delta=\frac{1}{1+\alpha}$,

$$|l_1(s)|\le C\frac{s^{2\delta}}{(1+s^2)^2}.$$

Since α is not an integer, $2\delta-1\neq 0$, two fundamental solutions of the homogeneous equation (f_{21} and f_{22}) can be found in explicit form:

$$f_{21}(s)=\frac{(2\delta+1)s^{2\delta}+(2\delta-1)s^{2\delta+2}}{1+s^2}$$
$$f_{22}(s)=\frac{(2\delta+1)s^{2-2\delta}+(2\delta-1)s^{-2\delta}}{1+s^2}$$

Let

$$w_1(s):=f_{21}(s)f_{22}'(s)-f_{21}'(s)f_{22}(s)=4\delta(1-4\delta^2)s^{-1}$$

and

$$f_1(s) = -\int_s^\infty \frac{f_{22}(\tau)l_1(\tau)}{w_1(\tau)} d\tau f_{21}(s) + \int_0^s \frac{f_{21}(\tau)l_1(\tau)}{w_1(\tau)} d\tau f_{22}(s),$$

then it is straightforward to verify

$$|f_1(s)| \le C\frac{s^{2\delta}}{(1+s)^3}, \quad 0 < s < \infty.$$

(3.1) is established. $\square$

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